The *De Continuo* of Thomas Bradwardine

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History or philsophy?

- Etienne Gilson, in *Being and Some Philosophers*:
  - [The] author may well have committed historical mistakes; he has not committed the deadly one of mistaking philosophy for history.
  - [P]hilosophy must choose; and applying to history for reasons to make a choice is no longer history, it is philosophy.

Thomas Bradwardine

- From C. S. Peirce:
  - Bradwardine "anticipated and outstripped our most modern mathematico-logicians, and gave the true analysis of continuity."
- Background:
  - Born around 1290 in Sussex
  - 1321 - 1323: Fellow at Balliol College, Oxford University
  - 1323: Fellow, and later the proctor, of Merton College
  - 1337: Chancellor of St. Paul's in London
  - 1339: Chaplain and confessor to Edward III
  - 19 June 1349: Archbishop of Canterbury
  - Died, 26 August 1349

Mention in Chaucer

- At least two references to Bradwardine in Chaucer:
  - Explicitly in the *The Nun's Priest's Tale*:

    In this matter, and great disputation,
    And has been (disputed) by a hundred thousand men.
    But I can not separate the valid and invalid arguments
    As can the holy doctor Augustine,
    Or Boethius, or the Bishop Bradwardyn,
    Whether God's worthy foreknowledge
    Constrains me by need to do a thing –
    "Need" I call simple necessity –
    Or else, if free choice be granted to me
    To do that same thing, or do it not,
    Though God knew it before I was born;
    Or if his knowledge constrains not at all
    But by conditional necessity.
    I will not have to do with such matter;

  - Implicitly in a discussion of the impossibility of dividing an indivisible in the *Summoner's Tale*.

Bradwardine's mathematical background

- Knows all 15 books of Euclid.
- Familiar with basic paradoxes of infinite sets:
  - For example, in the *De Causa Dei*,
    - ⋆ establishes a one-to-one correspondence between a denumerable set and an infinite fraction of itself.
    - ⋆ shows that the union of two denumerable sets is denumerable.
- One of the the Oxford Calculators
  - Example: worked with uniformly accelerated motion (discoveries sometimes attributed to Galileo).
- Wrote a well-known book on proportions, *Tractatus de proportionibus*.

Continua

- First of 24 definitions in the *De Continuo*
  - Continuum est quantum cuius partes ad invicem copulantur.
  - A continuum is a *quantum* whose parts are mutually joined to one another.
- A continuum may be geometrical (lines, surfaces, bodies), physical (space), or temporal (time, or motion).

Composition

- Definition 7:
  - Indivisibile est quod numquam dividi potest.
  - An indivisible is that which is never able to be divided.
- Definition 8:
  - Punctus est indivisibile situatum.
  - A point is an indivisible place.
- Question: Is a continuum composed from indivisibles?
- For example, is a line just a union of points?

Four answers

- No, a continuum is not composed from indivisibles.
- Yes, a continuum is composed of
  - a finite number of immediate indivisibles.
  - an infinite number of immediate indivisibles.
  - an infinite number of mediate indivisibles.
- The last of these is the standard modern answer, at least since the latter part of the 19^{th} century.

All the same?

- Question: Are all continua necessarily composed the same way?
- Bradwardine says they must be (Proposition 31).
- Proof: Given any continuum $\mathbb{X}$, a motion establishes a one-to-one correspondence between the indivisibles of $\mathbb{X}$ and the indivisibles of the time continuum.
- Hence all continua are composed in the same way as time.

All the same? (cont'd)

- Similarity with Peirce:
  - In *Law of Mind*, Peirce's argument for the reality of infinitesimals begins with time and consciousness:
    - [C]onsciousness must essentially cover an interval of time; for if it did not, we could gain no knowledge of time, and not merely no veracious cognition of it, but no conception whatever. We are, therefore, forced to say that we are immediately conscious through an infinitesimal interval of time.
  - And from the *Century Dictionary* of 1889:
    - [Continuous means] in mathematics and philosophy a connection of points (or other elements) as intimate as that of the instants or points of an interval of time: thus, the continuity of space consists in this, that a point can move from any one position to any other so that at each instant it shall have a definite and distinct position in space.

Indivisbles and lengths

- First of 10 suppositions:
  - Omne maius posse dividi in equale et in differentiam qua excedit.
  - Everything greater is able to be divided into a part equal and a difference by which it exceeds.
- First of 151 consquences:
  - Nullum indivisbile maius alio esse.
  - No indivisible is greater than another.
- It follows that, if line segments are composed from indivisibles, then two line segments composed from an equal number of indivisibles must have the same length.

Indivisibles and areas

- The 133rd conclusion follows:
  - Si sic, superficies composite ex lineis equalibus numero, longitudine sunt equales; si vero componantur ex lineis equalibus numero et inequalibus in longitudine, que ex longioribus componunter excedent; et idem de corporibus ex superficiebus compositis consequens esse scias.
- In modern terms:
  - Suppose A and B are planar sets, each a union of line segments:

$$A = \bigcup_{\omega \in \Omega_A} \ell_\omega \text{ and } B = \bigcup_{\omega \in \Omega_B} \ell_\omega.$$

  - Suppose each ℓ_ω is a line segment of finite length $m(\ell_\omega)$.
  - Moreover, suppose that $\varphi : \Omega_A \to \Omega_B$ is a one-to-one correspondence and

$$m(\ell_\omega) \geq m(\ell_{\varphi(\omega)})$$

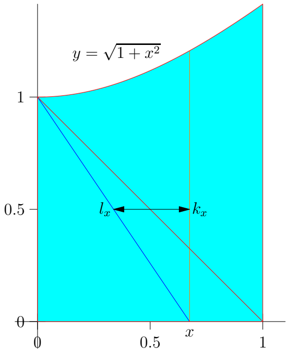
    for each $\omega \in \Omega_A$.
  - Then Bradwardine's conclusion is that

$$\text{Area of } A \geq \text{Area of } B.$$

An example

- Bradwardine shows how this conclusion leads to absurdities in a number of examples.
- For example, the 134$^{\text{th}}$ Conclusion states:
  - Si sic, omnis quadrati mediatas est maior toto quadrato.
  - If so, half of a square is greater than the whole square.
- His proof is basically as follows:
  - Let S the the unit square with vertices at $(0,0)$, $(0,1)$, $(1,1)$, and $(1,0)$.
  - Let T be the triangle with vertices at $(0,0)$, $(0,1)$, and $(1,0)$.
  - Let $\Omega_1 = \Omega_2 = [0,1]$.
  - For each $x \in [0,1]$, let ℓ_x be the line from $(x,0)$ to $(0,1)$, and let k_x be the line from $(x,0)$ to $(1,\sqrt{1+x^2})$.
  - Let $B = \bigcup_{x\in[0,1]} k_x$.
  - Now
  $$T = \bigcup_{x\in[0,1]} \ell_x \text{ and } m(\ell_x) = m(k_x)$$
  for all $x \in [0,1]$.

An example (cont'd)



- Hence, by Bradwardine's 133[rd] Conclusion,

$$\text{Area of } T = \text{Area of } B > \text{Area of } S = 2(\text{Area of } T).$$

Consquences

- Since the result is clearly false, either
  - the 133rd Conclusion is false, or
  - continua are not composed from indivisibles.
- Bradwardine chooses the second option.
- Connection with Adolf Grünbaum's interpretation ("A Consistent Conception of the Extended Linear Continuum as an Aggregate of Unextended Elements,", 1952) of a paradox of Zeno:
  - Grünbaum argued that the essence of Zeno's mistake is that measures are countably additive, but not uncountably additive.
  - But why are measures restricted to, at most, countable additivity?

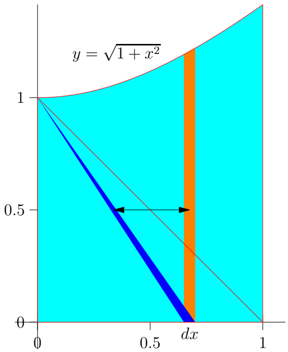
Bradwardine's option

- Corollary to Conclusion 141: A continuum is composed, not from indivisible parts, but from parts of the same type.
  - Omne continuum ex infinitis continuis similis speciei cum illo componi.
  - Every continuum is composed from an infinite number of continua of similar type with itself.
- That is, lines are composed from lines and planar regions from planar regions.
- An infinite number? What type of infinity?

Bradwardine's option (cont'd)

- Consider the one-to-one correspondence $\ell_x \leftrightarrow k_x$ as a transformation between two-dimension regions instead of as a transformation between lines.
- That is, starting with an infinitesimal part dx of the segment $[0, 1]$, consider the mapping which takes the triangle τ_{dx} with base dx and upper vertex at $(0, 1)$ to the region ρ_{dx} with base dx which extends up to the curve $y = \sqrt{1 + x^2}$.

Bradwardine's option (cont'd)



- Then T is composed from the infinitesimal triangles τ_{dx} and B is composed from the infinitesimal regions ρ_{dx}.

Bradwardine's option (cont'd)

- Moreover, since τ_{dx} has area $\frac{1}{2}dx$,

$$\text{Area of } T = \int_0^1 \frac{1}{2}dx = \frac{1}{2}.$$

- While

$$\text{Area of } B = \int_0^1 \sqrt{1+x^2}dx = \frac{1}{2}(1+\sinh^{-1}(1)) \approx 1.148.$$

- There is no contradiction because τ_{dx} and ρ_{dx} do not have the same area.

Surfaces, lines, and points?

- The final conclusion of the *De Continuo*:
  - Superficiem, lineam, sive punctum omnino non esse.
  - Surface, line, or point entirely not to be.
- That is, a line is not continuous by virtue of points being pasted together, nor is a line finite because of endpoints.
- That is, points are defined in relation to a line.
- Surfaces, lines, and points are *accidents* in the Aristotelian sense.
- Thomas Aquinas, (*In Metaphysica*, V, l. 9, 894):
  - Nam accidentis esse est inesse.
  - The *esse* of an accident is *inesse*.

Peirce, Kant, continuity

- Peirce (*Collected Papers*, 6.168):
  - [Kant] defines a continuum as that all of whose parts have parts of the same kind. He himself, and I after him, understood that to mean infinite divisibility, which plainly is not what constitutes continuity since the series of rational fractional values is infinitely divisible but is not by anybody regarded as continuous.
  - Kant's real definition implies that a continuous line contains no points. Now if we are to accept the common sense idea of continuity (after correcting its vagueness and fixing it to mean something) we must either say that a continuous line contains no points or we must say that the principle of excluded middle does not hold of these points.

Peirce, Kant, continuity (cont'd)

- And:
  - In the calculus and theory of functions it is assumed that between any two rational points (or points at distances along the line expressed by rational fractions) there are rational points and that further for every convergent series of such fractions (such as 3.1, 3.14, 3.141, 3.1415, 3.14159, etc.) there is just one limiting point; and such a collection of points is called continuous. But this does not seem to be the common sense idea of continuity. It is only a collection of independent points. Breaking grains of sand more and more will only make the sand more broken. It will not weld the grains into unbroken continuity.